B. Sc. Part II (Hons.). 4th Paper

Diff. Eqns (LDE CC)

Q. Find the complete integral of the equation $\frac{d^2x}{dt^2} + 2n\cos\alpha \frac{dx}{dt} + n^2x = a\cos nt$, which is such that when $t=0$, $x=0$ and $\frac{dx}{dt}=0$.

Soln Let $D \equiv \frac{d}{dt}$.

So, the given eqn is

$$D^2x + 2n\cos\alpha\, Dx + n^2x = a\cos nt$$

$$\Rightarrow [D^2 + 2n\cos\alpha\, D + n^2]x = a\cos nt$$

For CF $D^2 + 2n\cos\alpha\, D + n^2 = 0$

$$\Rightarrow D = \frac{-2n\cos\alpha \pm \sqrt{4n^2\cos^2\alpha - 4n^2}}{2}$$

$$\Rightarrow D = -n\cos\alpha \pm \sqrt{n^2(\cos^2\alpha - 1)}$$

$$\Rightarrow D = -n\cos\alpha \pm \sqrt{-n^2\sin^2\alpha}$$

$$\Rightarrow D = -n\cos\alpha \pm i\sin\alpha . n$$

$$\therefore CF = c_1 e^{(-n\cos\alpha + in\sin\alpha)t} + c_2 . e^{(-n\cos\alpha - i\sin\alpha . n)t}$$

$$= c_1 e^{-n\cos\alpha t} . e^{in\sin\alpha t} + c_2 e^{-n\cos\alpha t} . e^{-i\sin\alpha nt}$$

$\Rightarrow CF = e^{-nt\cos\alpha}\Big[c_1\{\cos(nt\sin\alpha) + i\sin(nt\sin\alpha)\}$

$+ c_2\{\cos(nt\sin\alpha) - i\sin(nt\sin\alpha)\}\Big]$

$\Rightarrow CF = e^{-nt\cos\alpha}\Big[(c_1+c_2)\cos(nt\sin\alpha) + i(c_1-c_2)\sin(nt\sin\alpha)\Big]$

$\Rightarrow CF = e^{-nt\cos\alpha}\left[A\cos(nt\sin\alpha) + B\sin(nt\sin\alpha)\right]$ — (1)

<u>For PI</u>

$$PI = \frac{1}{D^2 + (2n\cos\alpha)D + n^2}\, a\cos nt$$

$$= a\cdot\frac{1}{\cancel{-n^2} + 2n\cos\alpha D + \cancel{n^2}}\cos nt$$

$$= a\cdot\frac{1}{2n\cos\alpha}\int \cos nt\, dt$$

$$\Rightarrow PI = \frac{a}{2n\cos\alpha}\cdot\frac{\sin nt}{n} = \frac{a\sin nt}{2n^2\cos\alpha} \quad (2)$$

Hence, the complete soln is given by

$y = CF + PI$

where CF and PI are given by (1) and (2) respectively.